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Integrable modifications of Dicke and Jaynes–Cummings models, Bose–Hubbard dimers and classical *r*-matrices

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Abstract
We consider quantum integrable systems associated with non-skew-symmetric $sl(2)$-valued classical r-matrices. For a special class of such r-matrices we construct a one-parametric family of integrable modifications of the 'two-level one-mode' Jaynes–Cummings–Dicke Hamiltonians, with non-uniform coupling constants, containing additional Kerr- and Stark-type nonlinearities. We also construct a family of integrable Bose–Hubbard-type dimers and a family of integrable models that unifies Bose–Hubbard- and Jaynes–Cummings–Dicke-type models and may be called a 'two-level, two-mode' Jaynes–Cummings–Dicke model or a 'spin generalization' of a Bose–Hubbard dimer. We diagonalize the constructed models with the help of the algebraic Bethe ansatz technique in any irreducible representation of the $sl(2)^{\oplus N}$ spin algebra.

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1. Introduction

The simplest model describing an interaction of a one mode of a radiation field with a molecule of N two-level atoms is characterized by the following spin-boson Hamiltonian [1]:

$$\hat{H}_D = w\hat{b}^\dagger\hat{b} + \sum_{k=1}^{N} g_k\left(\hat{b}\hat{S}_+^{(k)} + \hat{b}^\dagger\hat{S}_-^{(k)}\right) + \sum_{k=1}^{N}\epsilon_k\hat{S}_3^{(k)}, \tag{1}$$

where the Bose operators $\hat{b}, \hat{b}^\dagger$ describe the field of radiation, the components of $sl(2)$-spins $\hat{S}_3^{(k)}, \hat{S}_\pm^{(k)}$ stand for the variables of the kth atom with the energies $\pm\frac{\epsilon_k}{2}$ and g_k is the strength of interaction of the field of radiation with the kth atom.

In the case $N = 1$ and the representation of $sl(2)$ with the highest weight $\lambda = \frac{1}{2}$, the Dicke model was solved by Jaynes and Cummings [2]. In the totally uniform case $g_k \equiv g, \epsilon_k \equiv \epsilon,\ k \in \overline{1, N}$, where N is arbitrary, the Dicke model was solved by Tavis and Cummings [3]. The more complicated case of the Dicke model with the equal interaction strength: $g_k \equiv g, k \in \overline{1, N}$, different energies $\epsilon_k \neq \epsilon_l, k, l \in \overline{1, N}$, and an arbitrary irreducible representation of the $sl(2)^{\oplus N}$ algebra was considered and exactly solved by Gaudin [4] (see also [5–7]).

In quantum optics, except for the simplest Hamiltonian (1), its special modifications that include additional nonlinear terms are frequently used. One of the most important of them is the following Hamiltonian:

$$\begin{aligned}\hat{H}_{mD} &= \hat{H}_D + V_{\mathrm{St}} + V_{\mathrm{Kerr}} \\ &= w\hat{b}^\dagger\hat{b} + \sum_{k=1}^{N} g_k\big(\hat{b}\hat{S}_+^{(k)} + \hat{b}^\dagger\hat{S}_-^{(k)}\big) + \sum_{k=1}^{N}\epsilon_k\hat{S}_3^{(k)} + k_1(\hat{b}^\dagger\hat{b})\sum_{k=1}^{N}\hat{S}_3^{(k)} + k_2(\hat{b}^\dagger\hat{b})^2, \end{aligned} \tag{2}$$

which includes coupling with a Kerr-like medium V_{Kerr} [8, 9] and Stark-shift term V_{St} [10].

In the present paper we consider the question of the complete quantum integrability of the totally non-uniform Hamiltonian (2) with $\epsilon_k \neq \epsilon_l, g_k \neq g_l,\ k, l \in \overline{1, N}$, in an arbitrary irreducible representation of the $sl(2)^{\oplus N}$ spin algebra. It is interesting to note that for $N > 1$ in an arbitrary representation of $sl(2)^{\oplus N}$ a general totally non-uniform Dicke Hamiltonian (1) is not integrable, but the more complicated 'modified' Hamiltonian (2) is (in some cases) integrable. More precisely, we show that there exists a one-parametric family of completely integrable Hamiltonians (2) of the following explicit form:

$$\begin{aligned}\hat{H}'_{mD} &= w_2\hat{b}_2^\dagger\hat{b}_2 + \sum_{k=1}^{N} g_k\big(\hat{b}_2\hat{S}_+^{(k)} + \hat{b}_2^\dagger\hat{S}_-^{(k)}\big) + \sum_{k=1}^{N}\epsilon_k\hat{S}_3^{(k)} \\ &\quad + (1-c_0)\hat{b}_2^\dagger\hat{b}_2\sum_{k=1}^{N}\hat{S}_3^{(k)} + \left(c_0 - \frac{1}{2}\right)\big(\hat{b}_2^\dagger\hat{b}_2\big)^2, \end{aligned} \tag{3}$$

where $w_2 = w - cc_0, \epsilon_k = -\big(2g_k^2 + w_2 + c\big)$ and the parameters w, c_0, c and g_k are arbitrary. Note that in the case $c_0 = \frac{1}{2}$ the Hamiltonian (3) contains only the Stark-shift term and in the case $c_0 = 1$ it contains only Kerr nonlinearity.

We also show that there exists the integrable 'two-level, two-mode' modified Dicke Hamiltonian, i.e. the Hamiltonian of a system of two bosons, interacting with N spins, having the following form:

$$\begin{aligned}\hat{H}_{dD} &= w_1\hat{b}_1^\dagger\hat{b}_1 + w_2\hat{b}_2^\dagger\hat{b}_2 + 4c_1c_2\big(\hat{b}_1^\dagger\hat{b}_2 + \hat{b}_2^\dagger\hat{b}_1\big) \\ &\quad + \sum_{k=1}^{N}\Big(\big(c_1\hat{b}_1g_k^{-1} - c_2\hat{b}_2g_k\big)\hat{S}_+^{(k)} + \big(c_1\hat{b}_1^\dagger g_k^{-1} - c_2\hat{b}_2^\dagger g_k\big)\hat{S}_-^{(k)}\Big) \\ &\quad + \sum_{k=1}^{N}\epsilon_k\hat{S}_3^{(k)} + \frac{1}{2}\left(c_0\left(\sum_{k=1}^{N}\hat{S}_3^{(k)} - \hat{b}_2^\dagger\hat{b}_2\right) - (c_0-1)\hat{b}_1^\dagger\hat{b}_1\right)^2 \\ &\quad + \frac{1}{2}\left((c_0-1)\left(\sum_{k=1}^{N}\hat{S}_3^{(k)} - \hat{b}_1^\dagger\hat{b}_1\right) - c_0\hat{b}_2^\dagger\hat{b}_2\right)^2, \end{aligned} \tag{4}$$

where $w_1 = 2(w - (c_0-1)c), w_2 = 2(w - c_0c), \epsilon_k = -\big(2c_2^2g_k^2 + 2c_1^2g_k^{-2} + 2w - c(2c_0-1)\big)$ and the parameters w, c_0, c_1, c_2, c and g_k are arbitrary. The Hamiltonian $\hat{H}_{mD}$ is recovered from the Hamiltonian $\hat{H}_{dD}$ in one boson case $(c_1 = 0, \hat{b}_1^\dagger\hat{b}_1 = 0)$, after subtracting from it the integrals of motion $(w - c_0 c)\hat{M}$ and $\frac{1}{2}\big(c_0^2 + (c_0 - 1)^2\big)\hat{M}^2$, where $\hat{M}$ is the operator of a number of excitations.

As a by-product of the model with the Hamiltonian (4) in the case $N = 0$ we obtain a one-parametric family of integrable modifications of Bose–Hubbard dimer models with the following Hamiltonian:

$$\hat{H}_{BH} = w_1 \hat{b}_1^\dagger \hat{b}_1 + w_2 \hat{b}_2^\dagger \hat{b}_2 + 4c_1 c_2 \big(\hat{b}_1^\dagger \hat{b}_2 + \hat{b}_2^\dagger \hat{b}_1\big) + \big((1 - c_0)\hat{b}_1^\dagger \hat{b}_1 - c_0 \hat{b}_2^\dagger \hat{b}_2\big)^2. \tag{5}$$

In the case $c_0 = \frac{1}{2}$ it coincides with the known integrable Bose–Hubbard Hamiltonian [11, 12].

Our technique is based on an application of the algebra of Lax operators and classical r-matrices. In more details, we develop our previous idea [18–23] that one can associate quantum integrable systems not only with skew-symmetric classical r-matrices but also with non-skew-symmetric ones. If, moreover, this r-matrix is 'diagonal' [20] one can also diagonalize the corresponding integrable Hamiltonians by means of the algebraic Bethe ansatz. Let us emphasize that contrary to skew-symmetric r-matrices, general non-skew-symmetric r-matrices are not connected with quantum groups or related structures. That is why in a general case our results cannot be obtained using a quantum group technique.

In the present paper we consider a class of examples of non-skew-symmetric classical r-matrices $r^c(u, v)$ labeled by the parameter c_0 and corresponding quantum integrable systems. These r-matrices are the simplest possible generalizations of the skew-symmetric trigonometric r-matrix and coincide with it in the special partial case $c_0 = \frac{1}{2}$. It is interesting to note that, although there exist many more complicated examples of non-skew-symmetric r-matrices [17–23] even the simplest of them produce new interesting quantum integrable systems. Indeed, in our previous papers [23, 24] we have used the r-matrices $r^c(u, v)$ in order to produce new integrable fermionic systems of reduced BCS-type. In the present paper we utilize these r-matrices in order to produce new integrable Dicke-type Hamiltonians containing additional Stark- and Kerr-type nonlinearities (2). In more detail, we show that among the Lax operators corresponding to these r-matrices there are Lax operators producing the Hamiltonians (3)–(5). We diagonalize these Hamiltonians and the commutative algebra of the first integral by means of the algebraic Bethe ansatz. It is necessary to note that the obtained integrable Hamiltonians (3)–(4) also seem to be new in the more standard case of skew-symmetric trigonometric r-matrices (case $c_0 = \frac{1}{2}$). In this partial case the Hamiltonian (3) does not contain Kerr-type nonlinearity.

The structure of this paper is as follows: in the second section we explain general relations of the theory of classical non-skew-symmetric r-matrices and the theory of quantum integrable systems. In the third section we concentrate on concrete examples of classical (non-skew-symmetric) r-matrices and the corresponding quantum integrable systems, namely, on Bose–Hubbard-type dimers and two types of modified Dicke models.

2. Quantum integrable systems and classical r-matrices

2.1. Definitions and notations

Let $\mathfrak{g} = sl(2)$ be the Lie algebra of traceless 2×2 matrices over the field of complex numbers. Let $\{X_3, X_+, X_-\}$, be the root basis in $sl(2)$ with the commutation relations:

$$[X_3, X_\pm] = \pm X_\pm, \qquad [X_+, X_-] = 2X_3.$$

Definition 1. *A function of two complex variables $r(u_1, u_2)$ with values in the tensor square of the algebra $sl(2)$ is called a classical r-matrix if it satisfies the following 'generalized' classical Yang–Baxter equation [14–16]:*

$$[r_{12}(u_1, u_2), r_{13}(u_1, u_3)] = [r_{23}(u_2, u_3), r_{12}(u_1, u_2)] - [r_{32}(u_3, u_2), r_{13}(u_1, u_3)],$$

where $r_{12}(u_1, u_2) \equiv r(u_1, u_2) \otimes 1$, etc.

We will be interested only in the meromorphic r-matrices for which there exists a re-parametrization $u = u(s)$, $v = v(t)$ such that the following decomposition holds true:

$$r(u(s), v(t)) = \frac{\Omega}{s - t} + r_0(u(s), v(t)), \tag{6}$$

where $r_0(u(s), v(t))$ is a holomorphic function with values in $sl(2) \otimes sl(2)$, $\Omega \in sl(2) \otimes sl(2)$ is the tensor Casimir: $\Omega = \frac{1}{2}(X_+ \otimes X_- + X_- \otimes X_+) + X_3 \otimes X_3$.

Moreover, in the present paper we will mainly consider 'diagonal' in the root basis r-matrices of the following explicit form:

$$r(u, v) = \left(\tfrac{1}{2}r^-(u, v)X_+ \otimes X_- + \tfrac{1}{2}r^+(u, v)X_- \otimes X_+ + r^3(u, v)X_3 \otimes X_3\right). \tag{7}$$

2.2. Algebra of Lax operators

Using a classical r-matrix $r(u, v)$, it is possible to define the 'tensor' Lie bracket in the space of certain $sl(2)$-valued functions of u with the operator coefficients $\hat{L}(u) = \hat{L}^3(u)X_3 + \hat{L}^+(u)X_+ + \hat{L}^-(u)X_-$:

$$[\hat{L}_1(u), \hat{L}_2(v)] = [r_{12}(u, v), \hat{L}_1(u)] - [r_{21}(v, u), \hat{L}_2(v)], \tag{8}$$

where $\hat{L}_1(u) = \hat{L}(u) \otimes 1$, $\hat{L}_2(v) = 1 \otimes \hat{L}(v)$.

The Lie bracket (8) has the following simple but important property, giving one a possibility of constructing new quantum systems with a new Lax operator using known quantum subsystems with the known Lax operators.

Proposition 2.1. *Let the operators $\hat{L}^{(k)}(u)$, $k \in \overline{1, N}$, have Lie bracket (8) and $[\hat{L}^{(k)}(u), \hat{L}^{(l)}(u)] = 0, k, l \in \overline{1, N}$. Then the operator $\hat{L}(u) \equiv \sum_{k=1}^{N} \hat{L}^{(k)}(u)$ also satisfies the Lie bracket (8).*

For the case of diagonal r-matrices the commutation relations (8) written in the component form are as follows:

$$[\hat{L}^-(u), \hat{L}^3(v)] = -(r^3(u, v)\hat{L}^-(u) + r^-(v, u)\hat{L}^-(v)), \tag{9a}$$

$$[\hat{L}^+(u), \hat{L}_3(v)] = (r^3(u, v)\hat{L}^+(u) + r^+(v, u)\hat{L}^+(v)), \tag{9b}$$

$$[\hat{L}^+(u), \hat{L}^-(v)] = -\tfrac{1}{2}(r^-(u, v)\hat{L}^3(u) + r^+(v, u)\hat{L}^3(v)), \tag{9c}$$

$$[\hat{L}^3(u), \hat{L}^3(v)] = [\hat{L}^+(u), \hat{L}^+(v)] = [\hat{L}^-(u), \hat{L}^-(v)] = 0. \tag{9d}$$

The components of the Lax operator $\hat{L}^\alpha(u)$ depend on an auxiliary parameter u and non-commuting quantum dynamical variables. The form of this dependence is not arbitrary but agrees with a structure of an r-matrix. In the following sections we will explicitly consider several types of such dependencies.

2.3. Quantum integrals

In this subsection we will explain the connection of classical non-skew-symmetric r-matrices with quantum integrability. It was shown in our previous paper [20] that just like in the case of classical r-matrix Lie–Poisson brackets [13–15] the Lie bracket (8) leads to an algebra of mutually commuting quantum integrals.

Let us consider the following quadratic in generators of the Lax algebra operators:

$$\hat{\tau}(u) = \tfrac{1}{2}(\hat{L}^3(u))^2 + (\hat{L}^+(u)\hat{L}^-(u) + \hat{L}^-(u)\hat{L}^+(u)). \qquad (10)$$

In order to obtain quantum integrable systems one has to show that $[\hat{\tau}(u), \hat{\tau}(v)] = 0$. This equality does not follow directly from the classical Poisson commutativity of $\tau(u)$ and $\tau(v)$ with respect to the corresponding Lie–Poisson brackets because of the problem of ordering of quantum operators. Nevertheless, the following theorem holds true [20].

Theorem 2.1. *Let $\hat{L}(u)$ be the Lax operator satisfying the commutation relations (8). Assume that in some open region $U \times U \subset \mathbb{C}^2$ the function $r(u, v)$ is meromorphic and possesses decomposition (6). Then the operator-valued function $\hat{\tau}(u)$ is a generator of a commutative algebra, i.e.*

$$[\hat{\tau}(u), \hat{\tau}(v)] = 0.$$

In the case of the r-matrices diagonal in the $sl(2)$ basis, the algebraic structure of the algebra of the Lax matrices permits one to diagonalize $\hat{\tau}(u)$ for all operators $\hat{L}(u)$, satisfying the commutation relations (2.2), using the algebraic Bethe ansatz technique.

2.4. Algebraic Bethe ansatz

In this subsection we diagonalize the generating functions of the quantum integrals $\hat{\tau}(u)$ for the case of the diagonal r-matrices (7) possessing the regularity property (6) and for the corresponding algebra of Lax operators (2.2). The procedure is the same for the cases of all types of diagonal r-matrices and all types of Lax operators in the representations possessing a 'vacuum' vector. That is why we proceed purely algebraically without fixation of the concrete form of the Lax operator as a function of the spectral parameter.

In more details, let $\mathcal{H}$ be a space of an irreducible representation of the algebra of Lax operators. Let us assume that there exists a vacuum vector $|0\rangle \in \mathcal{H}$ such that

$$\hat{L}^3(u)|0\rangle = \Lambda_3(u)|0\rangle, \qquad \hat{L}^-(u)|0\rangle = 0, \qquad (11)$$

and the whole space $\mathcal{H}$ is generated by the action of $\hat{L}^+(u)$ on the vector $|0\rangle$.

Using the explicit form of the generating function (10) it is easy to show that the vector $|0\rangle$ is an eigenvector of the generating function of the quantum Hamiltonians:

$$\hat{\tau}(u)|0\rangle = \tfrac{1}{2}\big(\Lambda_3(u)^2 + \partial_u\Lambda_3(u) + \big(r_0^-(u, u) + r_0^+(u, u)\big)\Lambda_3(u)\big)|0\rangle,$$

where $\Lambda_3(u)$ is an eigenvector of $\hat{L}_3(u)$ and we have used the equality

$$[\hat{L}^-(u), \hat{L}^+(u)] = \tfrac{1}{2}\big(\partial_u\hat{L}_3(u) + \big(r_0^-(u, u) + r_0^+(u, u)\big)\hat{L}_3(u)\big).$$

Let us now construct other eigenvectors of $\hat{\tau}(u)$ using the Bethe ansatz technique.

The following theorem holds true [20].

Theorem 2.2. *Let the components $\hat{L}^{\pm}(u)$, $\hat{L}^3(u)$ of the quantum Lax operator $\hat{L}(u)$ satisfy commutation relations (9) and r-matrix $r(u, v)$ possess the decomposition (6) with $u = s$, $v = t$. Let us consider vectors of the Bethe type:*

$$|v_1 v_2 \ldots v_M\rangle = \hat{L}^+(v_1)\hat{L}^+(v_2)\ldots\hat{L}^+(v_M)|0\rangle,$$

where the complex parameters v_i satisfy the following Bethe-type equations:

$$\Lambda_3(v_i) - \sum_{j=1, j\neq i}^{M} r^3(v_j, v_i) = r_0^3(v_i, v_i) - \frac{1}{2}(r_0^+(v_i, v_i) + r_0^-(v_i, v_i)), \qquad i \in 1, \ldots, M. \tag{12}$$

Then the vectors $|v_1 v_2 \ldots v_M\rangle$ are eigenvectors of the generating function of the quantum Hamiltonians $\hat{\tau}(u)$: $\hat{\tau}(u)|v_1 v_2 \ldots v_M\rangle = \Lambda(u|\{v_i\})|v_1 v_2 \ldots v_M\rangle$ with the following eigenvalues:

$$2\Lambda(u|\{v_i\}) = \left(\Lambda_3(u) - \sum_{i=1}^{M} r^3(v_i, u)\right)^2 - \sum_{i=1}^{M} r^+(v_i, u) r^-(v_i, u) + \left(r_0^+(u, u) + r_0^-(u, u)\right)\Lambda_3(u) + \partial_u \Lambda_3(u). \tag{13}$$

3. Dicke, Jaynes–Cummings and Bose–Hubbard-type models

3.1. *sl*(2)-'twisted' non-skew-symmetric r-matrices

Let us consider the non-skew-symmetric solution of the generalized classical Yang–Baxter equation on $sl(2)$ of the following explicit form (see [23]):

$$r_{12}^c(u, v) = \left(\frac{v^2}{u^2 - v^2} + c_0\right) X_3 \otimes X_3 + \frac{uv}{2(u^2 - v^2)}(X_+ \otimes X_- + X_- \otimes X_+), \tag{14}$$

where c_0 is an arbitrary constant.

It is evident that the considered r-matrix is diagonal in the $sl(2)$ basis and

$$r^{c,3}(u, v) = \frac{v^2}{u^2 - v^2} + c_0, \qquad r^{c,+}(u, v) = r^{c,-}(u, v) = \frac{uv}{(u^2 - v^2)}.$$

Remark 1. A parametrization in which the r-matrix possesses the decomposition (6) is the 'hyperbolic' parametrization $u^2 = e^s$, $v^2 = e^t$. For such a parametrization one has that

$$r_{12}(u(s), v(t)) = \frac{1}{s - t} X_3 \otimes X_3 + \frac{1}{2(s - t)}(X_+ \otimes X_- + X_- \otimes X_+) + r_{12}^0(s - t).$$

In this parametrization it is easy to show that $r_0^{c,3}(u, u) = c_0 - \frac{1}{2}$, $r_0^{c,+}(u, u) = r_0^{c,-}(u, u) = 0$.

Remark 2. The considered non-skew-symmetric r-matrix (14) is the simplest generalization of a skew-symmetric trigonometric r-matrix and coincides with it in the special case $c_0 = \frac{1}{2}$.

3.2. *Lax algebra and Bethe equations: general case*

Having fixed a concrete classical r-matrix it is possible to obtain more explicitly the Bethe equations and spectrum of the generating function of quantum integrals if the corresponding Lax algebra possesses a representation with the vacuum vector.

By direct calculation it is easy to obtain the more explicit form of the Bethe equations (12):

$$\Lambda_3(v_i) - \sum_{j=1, j\neq i}^{M} \frac{v_i^2}{v_j^2 - v_i^2} = M c_0 - \frac{1}{2}, \qquad i \in \overline{1, M}, \tag{15}$$

where $\Lambda_3(v_i)$ is an eigenvalue of the operator $\hat{L}^3(v_i)$ on the vacuum vector.

Using the Bethe equations (15) and concrete form of the non-skew-symmetric r-matrix it is possible to re-write formula (13) for the spectrum of the generating function of quantum integrals as follows:

$$\Lambda(u|\{v_i\}) = \frac{1}{2}(\Lambda_3(u) - M(c_0-1))^2 + \sum_{i=1}^{M} \frac{v_i^2(\Lambda_3(v_i) - \Lambda_3(u))}{v_i^2 - u^2} + \frac{1}{2}\partial_s \Lambda_3(u),$$

$$\text{where} \quad u^2 = e^s. \tag{16}$$

From this formula one deduces that $\Lambda(u|\{v_i\})$ does not have additional poles if $u = v_i$.

Now we will fix the form of the Lax operator in order to specialize the integrable model (i.e. its phase space, explicit form of Hamiltonian and quantum integrals) and in order to obtain a more explicit form of the Bethe equations (15) and a more explicit expression for the spectrum (16).

In the present paper we will be interested in special Lax operators $\hat{L}(u)$ possessing poles of an order not higher than one in the spectral parameter u^2.

The following proposition holds true.

Proposition 3.1. *Each of the following $sl(2)$-valued functions of u with operator coefficients*

$$\hat{L}^{(\infty)}(u) = \left(-c_0\hat{b}_2^\dagger\hat{b}_2 + c\right)X_3 + u\left(c_2\hat{b}_2X_+ + c_2\hat{b}_2^\dagger X_-\right) + 2u^2c_2^2X_3, \tag{17a}$$

$$\hat{L}^{(0)}(u) = \left((1-c_0)\hat{b}_1^\dagger\hat{b}_1 + c'\right)X_3 + u^{-1}\left(c_1\hat{b}_1X_+ + c_1\hat{b}_1^\dagger X_-\right) - u^{-2}2c_1^2X_3, \tag{17b}$$

$$L^{(g_i)}(u) = \left(\frac{u^2}{g_i^2 - u^2} + c_0\right)\hat{S}_3^{(i)}X_3 + \frac{ug_i}{2\left(g_i^2 - u^2\right)}\left(\hat{S}_+^{(i)}X_- + \hat{S}_-^{(i)}X_+\right) \tag{17c}$$

satisfies the linear r-matrix algebra (2.2). Here c, c', c_1, c_2, g_i, $i \in \overline{1,N}$, are the arbitrary constants, $\hat{b}_i^\dagger, \hat{b}_j, i, j \in \overline{1,2}$, constitute the Heisenberg algebra

$$\left[\hat{b}_i^\dagger, \hat{b}_j\right] = \delta_{ij}, \qquad \left[\hat{b}_i, \hat{b}_j\right] = \left[\hat{b}_i^\dagger, \hat{b}_j^\dagger\right] = 0, \tag{18}$$

and the operators $\hat{S}_\pm^{(i)}, \hat{S}_3^{(i)}, i \in \overline{1,N}$, constitute the spin $sl(2)^{\oplus N}$ algebra in some representation:

$$\left[\hat{S}_+^{(i)}, \hat{S}_-^{(j)}\right] = 2\delta_{ij}\hat{S}_3^{(i)}, \qquad \left[\hat{S}_+^{(i)}, \hat{S}_3^{(j)}\right] = -\delta_{ij}\hat{S}_+^{(i)}, \qquad \left[\hat{S}_-^{(i)}, \hat{S}_3^{(j)}\right] = \delta_{ij}\hat{S}_-^{(i)},$$
$$\left[\hat{S}_+^{(i)}, \hat{S}_+^{(j)}\right] = \left[\hat{S}_-^{(i)}, \hat{S}_-^{(j)}\right] = \left[\hat{S}_3^{(i)}, \hat{S}_3^{(j)}\right] = 0.$$

Remark 3. Note that the constants c, c' in the Lax operators (17*a*), (17*b*) play the role of the 'argument shift' [22]. It will be sufficient to keep only one of them, putting $c' = 0$.

Using the Lax operators constructed in this proposition and proposition 2.1 it is possible to construct from the 'elementary' Lax operators (17) the Lax operator of interacting systems. For example, if one considers the sum of the Lax operators (17*c*) for different points g_k, one arrives at the Lax operator of the Gaudin-type systems considered in our paper [23]. If one considers other combinations of the Lax operators (17*a*)–(17*c*), one comes to other quantum integrable systems. In the next subsection we will consider them in detail.

3.3. Example 1. Bose–Hubbard-type dimer

3.3.1. Lax operators and commuting integrals. Let us consider at first the simplest combination of the Lax matrices (3.1), namely

$$\hat{L}(u) = \hat{L}^{(\infty)}(u) + \hat{L}^{(0)}(u).$$

This will be the Lax operator of the Bose–Hubbard-type dimer. It has the following components:

$\hat{L}^3(u) = 2u^2c_2^2 + \big((1-c_0)\hat{b}_1^\dagger\hat{b}_1 - c_0\hat{b}_2^\dagger\hat{b}_2 + c\big) - 2u^{-2}c_1^2, \qquad \hat{L}^+(u) = uc_2\hat{b}_2 + u^{-1}c_1\hat{b}_1,$
$\hat{L}^-(u) = uc_2\hat{b}_2^\dagger + u^{-1}c_1\hat{b}_1^\dagger.$

Let us now consider the generating function $\hat{\tau}(u)$ and commutative integrals it produces via the decomposition $\hat{\tau}(u) = \sum_{k=-2}^{2} u^{2k}\hat{H}_{2k}$. By a direct calculation it is easy to show that $H_4 = 2c_2^4$, $H_{-4} = 2c_1^4$ and non-trivial Hamiltonians have the following form:

$$\hat{H}_0 = \tfrac{1}{2}\big((1-c_0)\hat{b}_1^\dagger\hat{b}_1 - c_0\hat{b}_2^\dagger\hat{b}_2 + c\big)^2 + 2c_1c_2\big(\hat{b}_1^\dagger\hat{b}_2 + \hat{b}_2^\dagger\hat{b}_1\big) - 4c_1^2c_2^2,$$
$$\hat{H}_2 = 2c_2^2\big((1-c_0)\big(\hat{b}_1^\dagger\hat{b}_1 + \hat{b}_2^\dagger\hat{b}_2\big) + c - \tfrac{1}{2}\big),$$
$$\hat{H}_{-2} = 2c_1^2\big(c_0\big(\hat{b}_1^\dagger\hat{b}_1 + \hat{b}_2^\dagger\hat{b}_2\big) - c - \tfrac{1}{2}\big).$$

The integrals $\hat{H}_2, \hat{H}_{-2}$ are proportional up to a constant to the operator of the number of particles $\hat{M} \equiv \hat{b}_1^\dagger\hat{b}_1 + \hat{b}_2^\dagger\hat{b}_2$. The generalized Bose–Hubbard dimer Hamiltonian obtained in the framework of our construction is $\hat{H}_{BH} = w\hat{M} + \hat{H}_0 + 4c_1^2c_2^2 - \frac{c^2}{2}$. It has has the form

$$\hat{H}_{BH} = w_1\hat{b}_1^\dagger\hat{b}_1 + w_2\hat{b}_2^\dagger\hat{b}_2 + 2c_1c_2\big(\hat{b}_1^\dagger\hat{b}_2 + \hat{b}_2^\dagger\hat{b}_1\big) + \tfrac{1}{2}\big((1-c_0)\hat{b}_1^\dagger\hat{b}_1 - c_0\hat{b}_2^\dagger\hat{b}_2\big)^2,$$

where $w_1 = w + c(1-c_0)$, $w_2 = w - cc_0$. In the partial case $c_0 = \frac{1}{2}$ it coincides with the standard Bose–Hubbard dimer Hamiltonian [12].

Remark 4. Note that by the re-parametrization of the spectral parameter u one can eliminate one of the coefficients c_i, $i \in 1, 2$, from a Lax operator, Hamiltonians, Bethe equations, etc. We have left both coefficients in all formulas in order to preserve symmetry between the bosons. Also note that it was necessary for the 'shift parameter' c to provide the independence of the frequencies w_1 and w_2 in the Hamiltonian $\hat{H}_{BH}$.

3.3.2. Spectrum. Let us now pass to a calculation of the spectrum of $\hat{\tau}(u)$ and $\hat{H}_i$. The representation space of the Heisenberg algebra (18) evidently admits the vacuum vector $|0\rangle$ such that

$$\hat{b}_i^\dagger|0\rangle = 0, \qquad i \in 1, 2, \qquad \text{and} \qquad \hat{L}^-(u)|0\rangle = 0.$$

It is easy to show that in this case we have $\hat{L}^3(u)|0\rangle = \Lambda_3(u)|0\rangle$, where

$$\Lambda_3(u) = -2u^{-2}c_1^2 + (1 - 2c_0 + c) + 2u^2c_2^2.$$

The eigenvalues of $\hat{t}(u)$ are calculated using formula (16) and have the following form:

$$\Lambda(u|\{v_i\}) = 2u^{-4}c_1^2 + u^{-2}h_{-2} + h_0 + u^2h_2 + 2u^4c_2^4,$$

where v_i satisfy Bethe-type equations obtained by a specialization of formula (15):

$$-2v_i^{-2}c_1^2 + \left(\frac{3}{2} + c - c_0(M+2)\right) + 2v_i^2c_2^2 = \sum_{j=1, j\neq i}^{M} \frac{v_i^2}{v_j^2 - v_i^2}, \qquad i \in \overline{1, M}. \tag{19}$$

By a direct calculation, specializing the general formula (16), one obtains the spectrum of $\hat{H}_0$ and $\hat{H}_{\pm 2}$:

$$h_0 = 2c_2^2\sum_{i=1}^{M} v_i^2 + \frac{1}{2}((1-c_0)(M+2) + c - 1)^2 - 4c_1^2c_2^2,$$
$$h_2 = 2c_2^2\left((1-c_0)(M+2) + c - \frac{1}{2}\right), \qquad h_{-2} = 2c_1^2\left((c_0(M+2) - c - \frac{1}{2}\right),$$

where v_i, $i \in \overline{1, M}$, are the solutions of equations (19) and $(M+2)$ is an eigenvalue of $\hat{M}$.

3.4. Example 2. Modified Dicke model

3.4.1. Lax operators and commuting integrals. Let us now consider more complicated Lax matrices namely the Lax matrices with $N+1$ simple poles in u^2 of the following form:

$$\hat{L}(u) = \hat{L}^{(\infty)}(u) + \sum_{k=1}^{N} \hat{L}^{(g_k)}(u).$$

As we will show below, this Lax matrix will correspond to the integrable Dicke-type spin-boson model. It has the following components:

$$\hat{L}^3(u) = \left((c_0-1)\sum_{k=1}^{N}\hat{S}_3^{(k)} - c_0\hat{b}_2^{\dagger}\hat{b}_2 + c\right) + 2c_2^2u^2 + \sum_{k=1}^{N}\frac{g_k^2\hat{S}_3^{(k)}}{g_k^2-u^2}, \tag{20}$$

$$\hat{L}^+(u) = u\left(c_2\hat{b}_2 + \sum_{k=1}^{N}\frac{g_k\hat{S}_-^{(k)}}{2\left(g_k^2-u^2\right)}\right), \qquad \hat{L}^-(u) = u\left(c_2\hat{b}_2^{\dagger} + \sum_{k=1}^{N}\frac{g_k\hat{S}_+^{(k)}}{2\left(g_k^2-u^2\right)}\right). \tag{21}$$

Let us consider the generating function $\hat{\tau}(u)$ and the commutative integrals it produces via the decomposition

$$\hat{\tau}(u) = \sum_{k=0}^{2} u^{2k}\hat{H}_{2k} + \sum_{k=0}^{N}\frac{g_k^2}{g_k^2-u^2}\hat{H}_{g_k} + \sum_{k=0}^{N}\frac{g_k^4}{\left(u^2-g_k^2\right)^2}\hat{C}_{g_k}.$$

Using the explicit expression for the components of the Lax operators (20) it is easy to show that $\hat{C}_{g_k}$ coincides with a Casimir operator of the kth copy of the $sl(2)$ Lie algebra:

$$\hat{C}_{g_k} = \tfrac{1}{2}\left(\left(\hat{S}_3^{(k)}\right)^2 + \tfrac{1}{2}\left(\hat{S}_+^{(k)}\hat{S}_-^{(k)} + \hat{S}_-^{(k)}\hat{S}_+^{(k)}\right)\right), \qquad k \in \overline{1,N}.$$

Other non-trivial integrals are

$$\hat{H}_{g_k} = c_2g_k\left(\hat{b}_2\hat{S}_+^{(k)} + \hat{b}_2^{\dagger}\hat{S}_-^{(k)}\right) + 2c_2^2g_k^2\hat{S}_3^{(k)} + \frac{1}{2}\left(\hat{S}_3^{(k)}\right)^2 + \left((c_0-1)\sum_{l=1}^{N}\hat{S}_3^{(l)} - c_0\hat{b}_2^{\dagger}\hat{b}_2 + c\right)\hat{S}_3^{(k)}$$
$$+ \sum_{l=1,l\neq k}^{N}\left(\frac{2g_l^2\hat{S}_3^{(k)}\hat{S}_3^{(l)} + g_kg_l\left(\hat{S}_+^{(k)}\hat{S}_-^{(l)} + \hat{S}_-^{(k)}\hat{S}_+^{(l)}\right)}{2\left(g_l^2-g_k^2\right)}\right) - \hat{C}_{g_k},$$

$$\hat{H}_0 = -c_2\left(\hat{b}_2\sum_{k=1}^{N}g_k\hat{S}_+^{(k)} + \hat{b}_2^{\dagger}\sum_{k=1}^{N}g_k\hat{S}_-^{(k)}\right)$$
$$- 2c_2^2\sum_{k=1}^{N}g_k^2\hat{S}_3^{(k)} + \frac{1}{2}\left((c_0-1)\sum_{k=1}^{N}\hat{S}_3^{(k)} - c_0\hat{b}_2^{\dagger}\hat{b}_2 + c\right)^2,$$

$$\hat{H}_2 = 2c_2^2\left((c_0-1)\left(\sum_{k=1}^{N}\hat{S}_3^{(k)} - \hat{b}_2^{\dagger}\hat{b}_2\right) + c - \frac{1}{2}\right) = 2c_2^2\left((1-c_0)\hat{M} + c - \frac{1}{2}\right),$$

where the integral $\hat{M} \equiv \hat{b}_2^{\dagger}\hat{b}_2 - \sum_{k=1}^{N}\hat{S}_3^{(k)}$ is the operator of the number of excitations.

Not all of the above Hamiltonians are independent. By direct calculation one obtains

$$\sum_{k=1}^{N}(\hat{H}_{g_k} + \hat{C}_{g_k}) + \hat{H}_0 = \frac{1}{2}\left(c_0\left(\sum_{k=1}^{N}\hat{S}_3^{(k)} - \hat{b}_2^{\dagger}\hat{b}_2\right) + c\right)^2 = \frac{1}{2}(c_0\hat{M}+c)^2.$$

One may take for the modified Dicke Hamiltonian the linear combination of the Hamiltonians $\hat{M}$ and $\hat{H}_0$: $\hat{H}_{mD} = w\hat{M} + \hat{H}_0 + \frac{c^2}{2}$ and obtain the following Dicke-type Hamiltonian:

$$\hat{H}_{mD} = w_2\hat{b}_2^\dagger\hat{b}_2 + \sum_{k=1}^{N} g_k\big(\hat{b}_2\hat{S}_+^{(k)} + \hat{b}_2^\dagger\hat{S}_-^{(k)}\big) - \sum_{k=1}^{N}\big(c + 2g_k^2 + w_2\big)\hat{S}_3^{(k)} + \frac{1}{2}\left((c_0 - 1)\sum_{k=1}^{N}\hat{S}_3^{(k)} - c_0\hat{b}_2^\dagger\hat{b}_2\right)^2, \tag{22}$$

where $w_2 \equiv w - cc_0$ and we have put $c_2 = -1$ (this condition can always be achieved by re-parametrization of spectral parameters and parameters g_k) and the parameters w, c_0, c are arbitrary. The Hamiltonian (3) is obtained from the Hamiltonian (22) by subtracting the integral of motion $\frac{1}{2}(c_0 - 1)^2\hat{M}^2$ from it.

The Hamiltonian $\hat{H}_{mD}$ may be interpreted as a Hamiltonian of a one mode of an electromagnetic field interacting with a molecule of N two-level atoms with coupling constants g_k and detuning parameters $\Delta_k = -\big(2g_k^2 + c\big)$ dependent on a number k of atoms in a molecule. The last term in the Hamiltonian (22) is not present in the standard Dicke Hamiltonian. The presence of this term in the Hamiltonian $\hat{H}_{mD}$ is necessary in order to preserve the integrability of the model in the non-uniform case.

Remark 5. In the case $N = 1$, the Hamiltonian (22) coincides with the modification of the Jaynes–Cummings Hamiltonian and has the following simple form:

$$\hat{H}_{mJC} = w_2\hat{b}_2^\dagger\hat{b}_2 + g\big(\hat{b}_2\hat{S}_+ + \hat{b}_2^\dagger\hat{S}_-\big) - (2g^2 + w_2 + c)\hat{S}_3 + \tfrac{1}{2}\big((c_0 - 1)\hat{S}_3 - c_0\hat{b}_2^\dagger\hat{b}_2\big)^2.$$

3.4.2. Spectrum. Let us now pass to a calculation of the spectrum of $\hat{\tau}(u)$ and $\hat{H}_i$. For this purpose it is necessary to consider a representation of the Lax algebra in some Hilbert space $\mathcal{H}$. In our case the Lax algebra is isomorphic to the algebra $H \oplus sl(2)^{\oplus N}$, where H is a Heisenberg algebra generated by the elements $b_2^\dagger$, b_2. Due to the fact that any irreducible representation of a direct sum of the Lie algebras is a tensor product of irreducible representations of its components, we will have $\mathcal{H} = V \otimes V^{\lambda_1} \otimes V^{\lambda_2} \otimes \cdots \otimes V^{\lambda_N}$, where V^{λ_k} is an irreducible finite-dimensional representation of the kth copy of $sl(2)$ with the spin λ_k, $\lambda_k \in \frac{1}{2}\mathbb{N}$, and V is an irreducible representation of the Heisenberg algebra H. Each of the representations V^{λ_k} contains the highest weight vector v_{λ_k} such that

$$\hat{S}_+^k\mathrm{v}_{\lambda_k} = 0, \qquad \hat{S}_3^k\mathrm{v}_{\lambda_k} = \lambda_k\mathrm{v}_{\lambda_k} \qquad \text{and} \qquad \hat{b}_2^\dagger\mathrm{v}_2 = 0,$$

where v_2 is the 'highest vector' of the Heisenberg algebra in the space V. Let us now consider the following 'vacuum' vector in the space $\mathcal{H}$: $|0\rangle = \mathrm{v}_2 \otimes \mathrm{v}_{\lambda_1} \otimes \cdots \otimes \mathrm{v}_{\lambda_N}$. We have

$$\hat{L}^-(u)|0\rangle = 0, \qquad \hat{L}^3(u)|0\rangle = \Lambda_3(u)|0\rangle,$$

where

$$\Lambda_3(u) = \left(c_0\sum_{k=1}^{N}\lambda_k + (c - c_0) + 2c_2^2u^2 + \sum_{k=1}^{N}\frac{\lambda_k u^2}{g_k^2 - u^2}\right)$$

by the very definition of $\hat{L}^-(u)$ and $\hat{L}^3(u)$.

By a direct calculation we obtain the following explicit form of the Bethe equations (15):

$$c_0\left(\sum_{l=1}^{N}\lambda_l - M - 1\right) + c + \frac{1}{2} + 2c_2^2v_i^2 + \sum_{l=1}^{N}\frac{\lambda_l v_i^2}{g_l^2 - v_i^2} = \sum_{j=1, j\neq i}^{M}\frac{v_i^2}{v_j^2 - v_i^2}, \qquad i \in \overline{1, M}. \tag{23}$$

It is possible to write the formula for the spectrum of the generating function of quantum integrals (13) in this case in the following explicit form:

$$\Lambda(u|\{v_i\}) = \sum_{k=0}^{2} u^{2k} h_{2k} + \sum_{k=0}^{N} \frac{g_k^2}{g_k^2 - u^2} h_{g_k} + \sum_{k=0}^{N} \frac{g_k^4}{\left(u^2 - g_k^2\right)^2} c_{g_k},$$

where

$$h_4 = 2c_2^4, \qquad c_{g_k} = \frac{1}{2}\lambda_k(\lambda_k + 1), \qquad h_2 = 2c_2^2\left((c_0 - 1)\left(\sum_{l=1}^{N}\lambda_l - M\right) + (c - c_0) + \frac{1}{2}\right),$$

$$h_0 = \frac{1}{2}\left((c_0 - 1)\left(\sum_{l=1}^{N}\lambda_l - M\right) + (c - c_0)\right)^2 + 2c_2^2\sum_{i=1}^{M} v_i^2 - 2c_2^2\sum_{l=1}^{N}\lambda_l g_l^2,$$

$$h_{g_k} = \lambda_k\left(\sum_{i=1}^{M}\frac{v_i^2}{g_k^2 - v_i^2} + \sum_{l=1, l\neq k}^{N}\frac{\lambda_l g_l^2}{g_l^2 - g_k^2} + 2c_2^2 g_k^2 + (c_0 - 1)\left(\sum_{l=1}^{N}\lambda_l - M\right) + (c - c_0) - \frac{1}{2}\right).$$

Here v_i^2, $i \in \overline{1, N}$, are the solutions of (23) and $(M + 1 - \sum_{l=1}^{N}\lambda_l)$ is a spectrum of $\hat{M}$.

3.5. *Example 3. 'Two-level, two-boson' Dicke model*

3.5.1. Lax operators and commuting integrals. Let us now consider the most general Lax matrix that can be obtained using proposition 3.1, namely the Lax operators of the following explicit form:

$$\hat{L}(u) = \hat{L}^{(\infty)}(u) + \hat{L}^{(0)}(u) + \sum_{k=1}^{N}\hat{L}^{(g_k)}(u).$$

It has the following components:

$$\hat{L}^3(u) = 2c_2^2u^2 - 2c_1^2u^{-2} + \left((1 - c_0)\hat{b}_1^\dagger\hat{b}_1 - c_0\hat{b}_2^\dagger\hat{b}_2 + c + (c_0 - 1)\sum_{k=1}^{N}\hat{S}_3^{(k)}\right) + \sum_{k=1}^{N}\frac{g_k^2\hat{S}_3^{(k)}}{g_k^2 - u^2},$$

$$\hat{L}^+(u) = \left(c_1\hat{b}_1u^{-1} + c_2\hat{b}_2u + \sum_{k=1}^{N}\frac{ug_k\hat{S}_-^{(k)}}{2\left(g_k^2 - u^2\right)}\right),$$

$$\hat{L}^-(u) = \left(c_1\hat{b}_1^\dagger u^{-1} + c_2\hat{b}_2^\dagger u + \sum_{k=1}^{N}\frac{ug_k\hat{S}_+^{(k)}}{2\left(g_k^2 - u^2\right)}\right).$$

Let us consider the generating function $\hat{\tau}(u)$ and the commutative integrals it produces via the decomposition

$$\hat{\tau}(u) = \sum_{k=-2}^{2} u^{2k}\hat{H}_{2k} + \sum_{k=0}^{N}\frac{g_k^2}{g_k^2 - u^2}\hat{H}_{g_k} + \sum_{k=0}^{N}\frac{g_k^4}{\left(u^2 - g_k^2\right)^2}\hat{C}_{g_k}.$$

It is easy to show that $\hat{C}_{g_k}$ coincides with a Casimir operator of the kth copy $sl(2)$ Lie algebra. Other trivial integrals are $\hat{H}_4 = 2c_2^4$, $\hat{H}_{-4} = 2c_1^4$. Non-trivial integrals have the form

$$\hat{H}_{-2} = -2c_1^2\left(c_0\left(\sum_{k=1}^{N}\hat{S}_3^{(k)} - \hat{b}_1^\dagger\hat{b}_1 - \hat{b}_2^\dagger\hat{b}_2\right) + c + \frac{1}{2}\right),$$

$$\hat{H}_2 = 2c_2^2\left((c_0 - 1)\left(\sum_{k=1}^{N}\hat{S}_3^{(k)} - \hat{b}_1^\dagger\hat{b}_1 - \hat{b}_2^\dagger\hat{b}_2\right) + c - \frac{1}{2}\right),$$

$$\hat{H}_{g_k} = c_1\big(g_k^{-1}b_1\hat{S}_+^{(k)} + g_k^{-1}b_1^\dagger\hat{S}_-^{(k)}\big) + c_2\big(g_k\hat{b}_2\hat{S}_+^{(k)} + g_k\hat{b}_2^\dagger\hat{S}_-^{(k)}\big) + 2c_2^2g_k^2\hat{S}_3^{(k)} - 2c_1^2g_k^{-2}\hat{S}_3^{(k)} + \frac{1}{2}\big(\hat{S}_3^{(k)}\big)^2 + \left((c_0-1)\sum_{k=1}^N\hat{S}_3^{(k)} - (c_0-1)\hat{b}_1^\dagger\hat{b}_1 - c_0\hat{b}_2^\dagger\hat{b}_2 + c\right)\hat{S}_3^{(k)} + \sum_{l=1,l\neq k}^N \frac{\big(2g_l^2\hat{S}_3^{(k)}\hat{S}_3^{(l)} + g_kg_l\big(\hat{S}_+^{(k)}\hat{S}_-^{(l)} + \hat{S}_-^{(k)}\hat{S}_+^{(l)}\big)\big)}{2\big(g_l^2-g_k^2\big)} - \hat{C}_{g_k},$$

$$\hat{H}_0 = 2c_1c_2\big(\hat{b}_1^\dagger\hat{b}_2 + \hat{b}_2^\dagger\hat{b}_1\big) - c_2\left(\hat{b}_2\sum_{k=1}^N g_k\hat{S}_+^{(k)} + \hat{b}_2^\dagger\sum_{k=1}^N g_k\hat{S}_-^{(k)}\right) - 2c_2^2\sum_{k=1}^N g_k^2\hat{S}_3^{(k)} + \frac{1}{2}\left((c_0-1)\sum_{k=1}^N\hat{S}_3^{(k)} - (c_0-1)\hat{b}_1^\dagger\hat{b}_1 - c_0\hat{b}_2^\dagger\hat{b}_2 + c\right)^2 - 4c_1^2c_2^2.$$

It is easy to see that the Hamiltonians $\hat{H}_{\pm 2}$ coincide up to non-important constants with the operator of a number of excitations: $\hat{M} \equiv \hat{b}_1^\dagger\hat{b}_1 + \hat{b}_2^\dagger\hat{b}_2 - \sum_{k=1}^N\hat{S}_3^{(k)}$. Using the Hamiltonians $\hat{H}_{g_k}$ it is possible to construct a counterpart of the Hamiltonian $\hat{H}_0$: $\hat{H}_0' = \sum_{k=1}^N(\hat{H}_{g_k} + \hat{C}_{g_k}) + \hat{H}_0$. It has the following form:

$$\hat{H}_0' = 2c_1c_2\big(\hat{b}_1^\dagger\hat{b}_2 + \hat{b}_2^\dagger\hat{b}_1\big) + c_1\left(\hat{b}_1\sum_{k=1}^N g_k^{-1}\hat{S}_+^{(k)} + \hat{b}_1^\dagger\sum_{k=1}^N g_k^{-1}\hat{S}_-^{(k)}\right) - 2c_1^2\sum_{k=1}^N g_k^{-2}\hat{S}_3^{(k)} + \frac{1}{2}\left(c_0\sum_{k=1}^N\hat{S}_3^{(k)} - (c_0-1)\hat{b}_1^\dagger\hat{b}_1 - c_0\hat{b}_2^\dagger\hat{b}_2 + c\right)^2 - 4c_1^2c_2^2.$$

Using the integrals $\hat{H}_0'$, $\hat{H}_0$ and number of particle operators $\hat{M}$ one constructs the 'two-level, two-mode' Dicke Hamiltonian (spin-dimer Hamiltonian) $\hat{H} = 2w\hat{M} + \hat{H}_0' + \hat{H}_0 + 8c_1^2c_2^2 - c^2$:

$$\hat{H} = w_1\hat{b}_1^\dagger\hat{b}_1 + w_2\hat{b}_2^\dagger\hat{b}_2 + 4c_1c_2\big(\hat{b}_1^\dagger\hat{b}_2 + \hat{b}_2^\dagger\hat{b}_1\big) + \sum_{k=1}^N\big(\big(c_1\hat{b}_1g_k^{-1} - c_2\hat{b}_2g_k\big)\hat{S}_+^{(k)} + \big(c_1\hat{b}_1^\dagger g_k^{-1} - c_2\hat{b}_2^\dagger g_k\big)\hat{S}_-^{(k)}\big) + \sum_{k=1}^N\epsilon_k\hat{S}_3^{(k)} + \frac{1}{2}\left(c_0\left(\sum_{k=1}^N\hat{S}_3^{(k)} - \hat{b}_2^\dagger\hat{b}_2\right) - (c_0-1)\hat{b}_1^\dagger\hat{b}_1\right)^2 + \frac{1}{2}\left((c_0-1)\left(\sum_{k=1}^N\hat{S}_3^{(k)} - \hat{b}_1^\dagger\hat{b}_1\right) - c_0\hat{b}_2^\dagger\hat{b}_2\right)^2,$$

where $w_1 = 2(w-(c_0-1)c)$, $w_2 = 2(w-c_0c)$ and $\epsilon_k = -\big(2c_2^2g_k^2+2c_1^2g_k^{-2}+2w-c(2c_0-1)\big)$.

3.5.2. Spectrum. Let us now pass to a calculation of the spectrum of $\hat{\tau}(u)$ and $\hat{H}_i$. For this purpose we will consider an irreducible representation of the Lax algebra in some Hilbert space $\mathcal{H}$. The Lax algebra in our case is isomorphic to the algebra $H^{\oplus 2}\oplus sl(2)^{\oplus N}$, where $H^{\oplus 2}$ is the Heisenberg algebra generated by the elements $\hat{b}_i^\dagger, \hat{b}_i, i\in\overline{1,2}$. Due to the fact that any irreducible representation of a direct sum of the Lie algebras is a tensor product of irreducible representations of its components, we will have $\mathcal{H} = V\otimes V^{\lambda_1}\otimes V^{\lambda_2}\otimes\cdots\otimes V^{\lambda_N}$, where V^{λ_k} is an irreducible finite-dimensional representation of the kth copy of $sl(2)$ with the spin λ_k, where $\lambda_k\in\frac{1}{2}\mathbb{N}$ and V is an irreducible representation of the Heisenberg algebra $H^{\oplus 2}$. Each representation V^{λ_k} contains a highest weight vector v_{λ_k} such that

$$\hat{S}_+^k\mathrm{v}_{\lambda_k} = 0, \qquad \hat{S}_3^k\mathrm{v}_{\lambda_k} = \lambda_k\mathrm{v}_{\lambda_k} \qquad \text{and} \qquad \hat{b}_i^\dagger\mathrm{v} = 0, \quad i\in\overline{1,2},$$

where v is the 'highest vector' of the Heisenberg algebra in the space V. Let us now consider the following 'vacuum' vector in the space $\mathcal{H}$: $|0\rangle = \mathrm{v}\otimes \mathrm{v}_{\lambda_1}\otimes\cdots\otimes \mathrm{v}_{\lambda_N}$. We have that

$$\hat{L}^-(u)|0\rangle = 0, \qquad \hat{L}^3(u)|0\rangle = \Lambda_3(u)|0\rangle,$$

where

$$\Lambda_3(u) = 2c_2^2u^2 - 2c_2^2u^2 + \sum_{k=1}^{N}\frac{\lambda_k g_k^2}{g_k^2-u^2} + (c_0-1)\sum_{k=1}^{N}\lambda_k + (c+1-2c_0)$$

due to the very definition of $\hat{L}^-(u)$ and $\hat{L}^3(u)$.

Using the explicit form of $\Lambda_3(u)$ we obtain the explicit form of the Bethe equations (15):

$$\left(c_0\left(\sum_{k=1}^{N}\lambda_k - M - 2\right) + c + \frac{3}{2}\right) + \left(2c_2^2v_i^2 - 2c_1^2v_i^{-2} + \sum_{k=1}^{N}\frac{\lambda_k v_i^2}{g_k^2-v_i^2}\right)$$
$$= \sum_{j=1, j\neq i}^{M}\frac{v_i^2}{v_j^2-v_i^2}, \qquad i\in\overline{1,M}.$$

By a direct calculation, using formula (16), it is possible to write explicitly the formula for the spectrum of the generating function of quantum integrals:

$$\Lambda(u|\{v_i\}) = \sum_{k=-2}^{2}u^{2k}h_{2k} + \sum_{k=0}^{N}\frac{g_k^2}{g_k^2-u^2}h_{g_k} + \sum_{k=0}^{N}\frac{g_k^4}{\left(u^2-g_k^2\right)^2}c_{g_k},$$

where $h_4 = 2c_2^4$, $h_{-4} = 2c_1^4$, $c_{g_k} = \frac{1}{2}\lambda_k(\lambda_k+1)$ and the eigenvalues of the non-trivial integrals are

$$h_{-2} = -2c_1^2\left(c_0\left(\sum_{k=1}^{N}\lambda_k - M - 2\right) + c + \frac{1}{2}\right),$$

$$h_2 = 2c_2^2\left((c_0-1)\left(\sum_{k=1}^{N}\lambda_k - M - 2\right) + c - \frac{1}{2}\right),$$

$$h_0 = 2c_2^2\sum_{i=1}^{M}v_i^2 - 2c_1^2\sum_{k=1}^{N}\lambda_k g_k^2 + \frac{1}{2}\left((c_0-1)\left(\sum_{k=1}^{N}\lambda_k - M - 2\right) + c - 1\right)^2 - 4c_1^2c_2^2,$$

$$h_{g_k} = \lambda_k\left(\sum_{i=1}^{M}\frac{v_i^2}{g_k^2-v_i^2} + \sum_{l=1, l\neq k}^{N}\frac{\lambda_l g_l^2}{g_l^2-g_k^2} + 2c_2^2g_k^2 - 2c_1^2g_k^{-2}\right.$$
$$\left. + (c_0-1)\left(\sum_{k=1}^{N}\lambda_k - M - 2\right) + c - \frac{3}{2}\right).$$

Here v_i^2 are the solutions of the Bethe equations, and $\left(M+2-\sum_{k=1}^{N}\lambda_k\right)$ is the eigenvalue of $\hat{M}$.

4. Conclusion and discussion

In the present paper we have constructed a one-parametric family of integrable modifications of the 'two-level one-mode' Jaynes–Cummings–Dicke Hamiltonians, a family of integrable Bose–Hubbard-type dimers and a family of integrable models that may be called 'two-level, two-mode' Jaynes–Cummings–Dicke models. We have calculated the spectrum of the quantum Hamiltonians of the proposed models using the algebraic Bethe ansatz.

It would be very interesting to find other physical quantities of these models. In particular it seems to be possible to construct correlation functions of the proposed models using *r*-matrix and Bethe ansatz techniques. For this purpose it is necessary to generalize the approach of [25] from the Gaudin to Dicke models and from the case of skew-symmetric *r*-matrices to non-skew-symmetric cases. We plan to return to this problem in our future publications.

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